

A Review on the Cosmology of the de Sitter Horndeski Models

Nelson J. Nunes¹ and Prado Martín-Moruno² and Francisco S. N. Lobo¹

¹*Instituto de Astrofísica e Ciências do Espaço, Universidade de Lisboa,
Faculdade de Ciências, Campo Grande, PT1749-016 Lisboa, Portugal*

²*Departamento de Física Teórica I, Universidad Complutense de Madrid, E-28040 Madrid*

We review the most general scalar-tensor cosmological models with up to second-order derivatives in the field equations that have a fixed spatially flat de Sitter critical point independent of the material content or vacuum energy. This subclass of the Horndeski Lagrangian is capable of dynamically adjusting any value of the vacuum energy of the matter fields at the critical point. We present the cosmological evolution of the linear models and the non-linear models with shift symmetry. We come to the conclusion that the shift symmetric non-linear models can deliver a viable background compatible with current observations.

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I. INTRODUCTION

The realisation that the Universe is currently undergoing an accelerated expansion is one of the major discoveries in cosmology. During the last eighteen years, a number of proposals to explain this evolution have been suggested. Most proposals involve scalar field dark energy (quintessence, k-essence, kinetic braiding) or extensions of Einstein's general relativity. These models are in principle stable, as their equations of motion are only second-order. However, Lagrangians consisting of second-order derivatives generally give rise to equations of motion with higher-order derivatives. Such theories might propagate a ghost degree of freedom, or in other words, they have an Ostrogradski instability [1]. In 1974, Horndeski wrote down the most general scalar-tensor theory leading to second-order equations of motion [2]. Despite being unnoticed for almost four decades, Deffayet et al. [3] rediscovered this theory when generalizing the covariantized version [4] of the galileons models [5]. It turns out that Brans–Dicke theory, k-essence, kinetic braiding, or $f(R)$ models are subclasses of the most general Horndeski Lagrangian. The theory can be written in terms of the arbitrary functions $\kappa_i(\phi, X)$ and $F(\phi, X)$, where $X = \partial_\mu \phi \partial^\mu \phi$. Thus, although the Horndeski theory restricts the type of stable scalar-tensor theories, there is still a huge amount of freedom.

As the vacuum energy gravitates in extensions to general relativity, the cosmological constant problem persists whenever the scalar field can only screen a given value of that constant [6–8]. In order to address this problem, Charmousis et al. [9, 10] introduced the “fab four” models. In these models, the scalar field may acquire a non-trivial time dependence once the cosmological constant has been screened, hence avoiding Weinberg's no-go theorem. This screening was constructed demanding that the critical point of the dynamics is Minkowski. However, also by construction, as the dynamics approaches Minkowski, the universe is forced to decelerate. Therefore, a universe accelerating at late time does not naturally arise in this set up. In this article, we review how the concept of self-adjustment was extended from Minkowskian to de Sitter final states [11] and show that these models can lead to very promising cosmological scenarios from the observational point of view [12–14].

A. Dynamical Screening

Let us consider a FLRW geometry of the universe. After integrating the higher derivatives by parts, the Horndeski Lagrangian can be written as [10]

$$L(a, \dot{a}, \phi, \dot{\phi}) = a^3 \sum_{i=0}^3 Z_i(a, \phi, \dot{\phi}) H^i, \quad \text{where} \quad L = V^{-1} \int d^3x \mathcal{L}_H, \quad (1)$$

$H = \dot{a}/a$ is the Hubble expansion rate, V is the spatial integral of the volume element, and a dot identifies a derivative with respect to the cosmic time t . The functions Z_i are written as

$$Z_i(a, \phi, \dot{\phi}) = X_i(\phi, \dot{\phi}) - \frac{k}{a^2} Y_i(\phi, \dot{\phi}), \quad (2)$$

where X_i and Y_i are given in terms of the Horndeski free functions [10]. The Hamiltonian density yields

$$\mathcal{H}(a, \dot{a}, \phi, \dot{\phi}) = \frac{1}{a^3} \left[\frac{\partial L}{\partial \dot{a}} \dot{a} + \frac{\partial L}{\partial \dot{\phi}} \dot{\phi} - L \right] = \sum_{i=0}^3 \left[(i-1) Z_i + Z_{i, \dot{\phi}} \dot{\phi} \right] H^i. \quad (3)$$

Let us assume that the matter fluids, given by the energy density $\rho_m(a)$, are minimally coupled and do not interact with the scalar field. The Friedmann equation is then obtained from

$$\mathcal{H} + \mathcal{H}_{\text{EH}} + \mathcal{H}_{\text{matter}} = 0, \quad (4)$$

where the Einstein–Hilbert Hamiltonian density is $\mathcal{H}_{\text{EH}} = -3M_{\text{Pl}}^2 H^2$ and the matter component is $\mathcal{H}_{\text{matter}} = \rho_m$. We will follow the same procedure described in Ref. [10] applied to Minkowski, but now requiring that self-tuning applies to a more general late-time solution or critical point with $H^2 \rightarrow H_c^2 \neq 0$. Ideally, we would like this solution H_c to be an attractor solution; however, this particular adjustment mechanism can only ensure that it is a critical point. The recipe for a successful screening mechanism is the following:

1. At the critical point, the field equation must be trivially satisfied such that the value of the scalar field is free to screen. This means that, up to a total derivative, at the critical point the Lagrangian density must be independent of both ϕ and $\dot{\phi}$

$$\sum_{i=0}^3 Z_i(a_c, \phi, \dot{\phi}) H_c^i = c(a_c) + \frac{1}{a_c^3} \frac{d\zeta(a_c, \phi)}{dt}. \quad (5)$$

This immediately shows that $\sum_i Z_i(a_c, \phi, \dot{\phi}) H_c^i$ is at most linear in $\dot{\phi}$.

2. In order to compensate for possible discontinuities of the cosmological constant appearing on the right hand side of the Friedmann equation, this equation must depend on $\dot{\phi}$ once screening has taken place. In other words, $\mathcal{H}_{,\dot{\phi}} \neq 0$. Taking into account Equation (4) and given that $\sum_i Z_{i,\dot{\phi}}(a_c, \phi, \dot{\phi}) H_c^i = 0$, as we saw above, it leads to

$$\sum_{i=1}^3 i Z_{i,\dot{\phi}}(a_c, \phi, \dot{\phi}) H_c^i \neq 0. \quad (6)$$

3. Requiring a non-trivial cosmology before screening implies that the scalar field equation of motion must depend on \dot{H} . This leads to the same condition (6) if $H_c \neq 0$. In other words, $Z_{i,\dot{\phi}}(a_c, \phi, \dot{\phi}) H_c^i \neq 0$ for at least one value of i .

Let us take a particular Lagrangian, \bar{L} , that satisfies these conditions at the critical point $a = a_c$,

$$\bar{L} = \sum_{i=0}^3 \bar{Z}_i(a_c, \phi, \dot{\phi}) H_c^i, \quad (7)$$

and

$$\sum_{i=1}^3 i \bar{Z}_{i,\dot{\phi}}(a_c, \phi, \dot{\phi}) H_c^i \neq 0, \quad (8)$$

as before, and where \bar{Z}_0 is arbitrary. We now choose \bar{Z}_0 such that at the critical point the Lagrangian is $\bar{L} = c(a_c)$. The Lagrangian is given quite generically as

$$\bar{L}(a, \dot{a}, \phi, \dot{\phi}) = a^3 \left[c(a) + \sum_{i=1}^3 \bar{Z}_i(a, \phi, \dot{\phi}) (H^i - H_c^i) \right]. \quad (9)$$

By construction, this Lagrangian has a critical point at H_c . We will now search for the form of the \bar{Z}_i 's. As it was explicitly shown in Ref. [10], two Horndeski Lagrangians which self-tune to H_c are related by a total derivative of a function $\mu(a, \phi)$, such that

$$L(a, \dot{a}, \phi, \dot{\phi}) = \bar{L}(a, \dot{a}, \phi, \dot{\phi}) + \frac{d\mu(a, \phi)}{dt}. \quad (10)$$

This relation must be valid during the whole evolution; therefore, equating equal powers of H , we obtain

$$Z_0 = c(a) - \sum_{i=1}^3 \bar{Z}_i H_c^i + \frac{\dot{\phi}}{a^3} \mu_{,\phi}, \quad Z_1 = \bar{Z}_1 + \frac{1}{a^2} \mu_{,a}, \quad Z_2 = \bar{Z}_2, \quad Z_3 = \bar{Z}_3, \quad (11)$$

which upon substituting \bar{Z}_i in the first of the above equations yields [10, 11]

$$\sum_{i=0}^3 Z_i(a, \phi, \dot{\phi}) H_c^i = c(a) + \frac{H_c}{a^2} \mu_{,a}(a, \phi) + \frac{\dot{\phi}}{a^3} \mu_{,\phi}(a, \phi). \quad (12)$$

B. The de Sitter Critical Point: $H_c^2 = \Lambda$

Let us first consider a flat universe with $k = 0$, which means that the dependence of Z_i 's on the scale factor and Y_i 's disappears. We also require that $H_c^2 = \Lambda$, which leads to

$$\sum_{i=0}^3 X_i(\phi, \dot{\phi}) \Lambda^{i/2} = c(a) + \frac{\sqrt{\Lambda}}{a^2} \mu_{,a} + \frac{\dot{\phi}}{a^3} \mu_{,\phi}. \quad (13)$$

As the left hand side of this equation is independent of a , so should the right hand side be for any value of $\dot{\phi}$. The function $\mu(a, \phi)$ must, therefore, be of the form

$$\mu(a, \phi) = a^3 h(\phi) - \frac{1}{\sqrt{\Lambda}} \int da c(a) a^2. \quad (14)$$

Thus, we have

$$L_c = a^3 \sum_{i=0}^3 X_i(\phi, \dot{\phi}) \Lambda^{i/2} = a^3 \left(3\sqrt{\Lambda} h(\phi) + \dot{\phi} h_{,\phi}(\phi) \right). \quad (15)$$

Therefore, we can consider three different kinds of terms in the Lagrangian. These are: (i) X_i -terms linear in $\dot{\phi}$; (ii) X_i -terms non-linear in $\dot{\phi}$, the contribution of which must vanish at the critical point; and, (iii) terms not able to self-tune as they contribute via total derivatives, or terms that multiply by the curvature k in the Lagrangian.

Let us now consider terms with an arbitrary dependence on ϕ and $\dot{\phi}$. The Lagrangian is given by

$$L = a^3 \sum_{i=0}^3 X_i(\phi, \dot{\phi}) H^i, \quad (16)$$

and the Hamiltonian density is

$$\mathcal{H} = \sum_{i=0}^3 \left[(i-1) X_i(\phi, \dot{\phi}) + \dot{\phi} X_{i,\dot{\phi}}(\phi, \dot{\phi}) \right] H^i. \quad (17)$$

The field equation can be written as

$$\sum_{i=0}^3 \left[X_{i,\phi} - 3X_{i,\dot{\phi}} H - iX_{i,\phi} \frac{\dot{H}}{H} - X_{i,\phi\phi} \dot{\phi} - X_{i,\phi\dot{\phi}} \ddot{\phi} \right] H^i = 0. \quad (18)$$

II. LINEAR MODELS “THE MAGNIFICENT SEVEN”

In order to satisfy Equation (15) considering only terms linear on $\dot{\phi}$, it is sufficient to set

$$X_i^{\text{ms}}(\phi, \dot{\phi}) = 3\sqrt{\Lambda} U_i(\phi) + \dot{\phi} W_i(\phi), \quad (19)$$

provided the potentials U_i and W_i satisfy the constraint

$$\sum_{i=0}^3 W_i(\phi) \Lambda^{i/2} = \sum_{i=0}^3 U_{i,\phi}(\phi) \Lambda^{i/2}. \quad (20)$$

As there are in total eight functions U_i and W_i , and only one constraint, there are seven free functions—the magnificent seven. In these models, the field equation and the Friedmann equation read [12],

$$H' = 3 \frac{\sum_i H^i \left(\sqrt{\Lambda} U_{i,\phi}(\phi) - H W_i(\phi) \right)}{\sum_i i H^i W_i(\phi)}, \quad \phi' = \sqrt{\Lambda} \frac{(1 - \Omega) H^2 - 3 \sum_i (i-1) H^i U_i(\phi)}{\sum_i i H^{i+1} W_i(\phi)}, \quad (21)$$

where a prime means a derivative with respect to $\ln a$. The critical point of the system is $(H_c, \phi_c, \Omega) = (\sqrt{\Lambda}, \phi_c, 0)$, and its stability depends on the particular form of U_i and W_i [12]. We are now going to consider a number of cases in our search for viable cosmological models compatible with current observations.

A. Only $W_0 \neq 0$

Let us first assume that $W_0 \neq 0$, and $W_1 = W_2 = W_3 = 0$. In this case, H' is ill-defined as the denominator of (21) vanishes. This can be understood by inspecting the Hamiltonian

$$\mathcal{H}_{\text{linear}} = \sum_i \left[3(i-1)\sqrt{\Lambda} U_i(\phi) + i \dot{\phi} W_i(\phi) \right] H^i = \sum_i \left[3(i-1)\sqrt{\Lambda} U_i(\phi) \right] H^i.$$

We see that this Hamiltonian is independent of $\dot{\phi}$, therefore violating condition (ii) for a successful Lagrangian. Thus, this model does not screen dynamically, and only the de Sitter solution exists.

B. Only a W_i, U_j Pair

From the constraint equation, we have that $W_i = U_{j,\phi} \Lambda^{(j-i)/2}$, and then

$$\frac{H'}{H} = -\frac{3}{i} \left[1 - \left(\frac{H}{\sqrt{\Lambda}} \right)^{j-i-1} \right],$$

which is independent of ϕ , and consequently, the matter content has no influence on the Universe's evolution. When $j - i - 1 < 0$, the de Sitter solution is an attractor. When $H \gg \sqrt{\Lambda}$, the field equation can be approximated by $H'/H = -3/i$. We can obtain a dust-like behaviour provided $i = 2$ and—as we expected by construction—we reach a de Sitter evolution when $H \rightarrow \sqrt{\Lambda}$.

C. Only a W_i, W_j Pair

In this case, from the constraint equation, $W_i = -W_{j,\phi} \Lambda^{(j-i)/2}$, and we have

$$\frac{H'}{H} = -3 \frac{1 - (H/\sqrt{\Lambda})^{i-j}}{j - i(H/\sqrt{\Lambda})^{i-j}},$$

which is again ϕ independent. For $j > i$, the de Sitter solution is an attractor. For $H \gg \sqrt{\Lambda}$, we can approximate the field equation as $H'/H = -3/j$, and we recover a dust-like evolution for $j = 2$. A de Sitter universe is attained when $H \rightarrow \sqrt{\Lambda}$.

D. Term-by-Term Model

We now consider that the constraint equation is satisfied for equal powers of Λ , such that $W_i = U_{i,\phi}$. There are eight functions and four constraints; therefore, only four free potentials. Defining $U_{i,\phi} = \Lambda^{-i/2} V_{i,\phi}$, we can write

$$\frac{H'}{H} = -3 \left(1 - \frac{\sqrt{\Lambda}}{H} \right) \frac{\sum_i (H/\sqrt{\Lambda})^i V_{i,\phi}}{\sum_i i (H/\sqrt{\Lambda})^i V_{i,\phi}}.$$

In this case, the scalar field contributes to the dynamics of the universe, as there is a dependence on ϕ , which is itself determined by the matter content via Equation (21). For $H \gg \sqrt{\Lambda}$ and when only one i component dominates, $H'/H = -3/i$, which means that dust is recovered for $i = 2$. As before, we reach de Sitter when $H \rightarrow \sqrt{\Lambda}$.

E. Tripod Model

Let us consider the three potentials U_2, U_3 , and W_2 . The constraint equation imposes $U_{2,\phi} \Lambda + U_{3,\phi} \Lambda^{3/2} = W_2 \Lambda$, and then

$$\frac{H'}{H} = -3 \frac{U_{2,\phi}}{W_2} \left(1 - \frac{\sqrt{\Lambda}}{H} \right).$$

For $H \gg \sqrt{\Lambda}$, we have approximately

$$\frac{H'}{H} = -\frac{3}{2} \frac{U_{2,\phi}}{W_2}.$$

In order to obtain a cosmological viable model, we need: $U_{2,\phi}/W_2 = 1$ during a matter domination epoch, and $U_{2,\phi}/W_2 = 4/3$ for a radiation domination epoch. This can be achieved with the choice of potentials, $U_2 = e^{\lambda\phi} + \frac{4}{3}e^{\beta\phi}$, and $W_2 = \lambda e^{\lambda\phi} + \beta e^{\beta\phi}$, as shown in Figure 1. The de Sitter evolution is obtained when $H \rightarrow \sqrt{\Lambda}$. Unfortunately, the

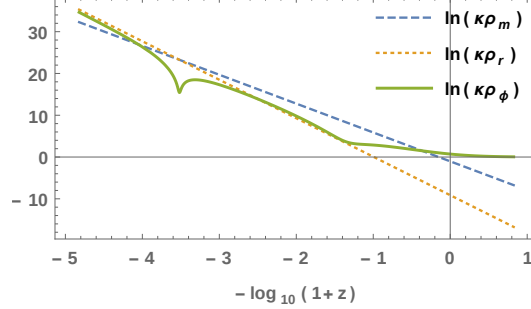


FIG. 1: The evolution of the energy densities for the tripod models. Figure from [12].

field has a large contribution at early time which is incompatible with current constraints.

III. NON-LINEAR MODELS

In this section we consider that $X_i(\phi, \dot{\phi})$ are non-linear terms in $\dot{\phi}$ in the Lagrangian

$$L_{nl} = a^3 \sum_{i=0}^3 X_i(\phi, \dot{\phi}) H^i. \quad (22)$$

As we saw before, any non-linear dependence of the Lagrangian on $\dot{\phi}$ must vanish at the critical point; thus, $\sum_{i=0}^3 X_i(\phi, \dot{\phi}) \Lambda^{i/2} = 0$. We will restrict the analysis to the shift-symmetric cases, as it simplifies the calculations. Moreover, these cases also lead to a radiative stable situation since the field is non-renormalizable [15]. Therefore, the system is independent of ϕ , and we will make use of the convenient redefinition, $\psi = \dot{\phi}$. Under these assumptions, the field equation and the Friedmann equation are [14],

$$H' = \frac{3(1+w)Q_0P_1 - Q_1P_0}{Q_1P_2 - Q_2P_1}, \quad \psi' = \frac{3(1+w)Q_0P_2 - Q_2P_0}{Q_2P_1 - Q_1P_2},$$

where $Q_0, Q_1, Q_2, P_0, P_1, P_2$, are complicated functions of X_i and H , and the average equation of state parameter of matter fluids is

$$1 + w = \frac{\sum_s \Omega_s (1 + w_s)}{\sum_s \Omega_s}.$$

The eigenvalues of the Jacobian matrix of the system formed by H' and ψ' evaluated at the critical point are $(-3, -3(1+w))$, which means that the critical point is stable whenever $w > -1$.

As for the linear models, we are now going to take a systematic evaluation of the possible cosmological scenarios. In what follows, we will redefine X_i such that $X_i = 3M_{Pl}^2 \Lambda^{1-i/2} f_i$.

A. $f_3 = \psi^n$ Is the Dominant Contribution

When f_3 is the dominant potential and $H \gg \sqrt{\Lambda}$, the effective equation of state is

$$1 + w_{\text{eff}} \simeq \frac{2}{3}(1+w), \quad \text{for} \quad \frac{|(2f_3 + \psi f_{3,\psi}) f_{3,\psi\psi}|}{|(3f_{3,\psi} + \psi f_{3,\psi\psi}) f_{3,\psi}|} \gg 1$$

$$1 + w_{\text{eff}} \simeq \frac{2}{3}, \quad \text{otherwise.}$$

Neither of these allow for w_{eff} corresponding to radiation and/or matter domination epochs.

B. $f_2 = \psi^n$ Is the Dominant Contribution

If instead f_2 is the dominant potential, for $H \gg \sqrt{\Lambda}$, it follows that

$$w_{\text{eff}} \simeq w, \quad \text{for} \quad \frac{|(1 - f_2 - \psi f_{2,\psi}) f_{2,\psi\psi}|}{|(2f_{2,\psi} + \psi f_{2,\psi\psi}) f_{2,\psi}|} \gg 1,$$

$$w_{\text{eff}} \simeq 0, \quad \text{otherwise.}$$

In this case, either w_{eff} is too small at present when compared with observational constraints, or Ω_ψ is too large in the early universe.

C. f_0 and f_1 Are the Sole Contributions

If we take f_0 and f_1 to be the only non-negligible potentials, then it can be shown that when $H \gg \sqrt{\Lambda}$, the equation of state parameter $w_{\text{eff}} \simeq w$. This represents an interesting case, but unfortunately, models with realistic initial conditions do not evolve to the critical point.

D. Extension with f_0 , f_1 and f_2

Finally, we consider a case involving the three potentials X_0 , X_1 , and X_2 , such that

$$f_2(\psi) = \alpha\psi^n, \quad f_1(\psi) = -\alpha\psi^n + \frac{\beta}{\psi^m}, \quad f_0(\psi) = -\frac{\beta}{\psi^m}.$$

None of the potentials dominates the whole evolution; instead, different potentials are important at different epochs. This is a very promising case in what regards a background behaviour. We can obtain a model with $w_\psi = w_0 + w_a(1-a)$, such that $w_0 = -0.98$ and $w_a = 0.04$, which is compatible with current observational bounds. Moreover, the example gives a negligible dark energy contribution at early times. The evolution of the energy densities of the field and matter fluids is illustrated in Figure 2.

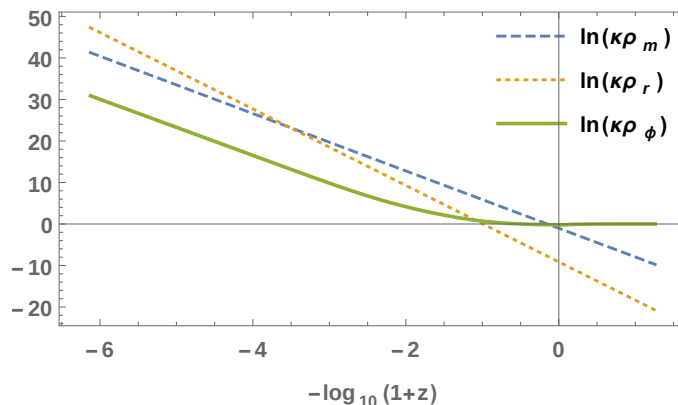


FIG. 2: The evolution of the energy densities for the model with non-negligible X_0 , X_1 , and X_2 . Figure from [14].

IV. SUMMARY

In this article, we have considered a subclass of the Horndeski cosmological models that may alleviate the cosmological constant problem by screening any value the vacuum energy might take. They lead to a final de Sitter evolution of the universe regardless of the matter content. We have presented linear and the non-linear models and shown

that the class of non-linear models with shift symmetry is very promising when tested against current observational constraints on the effective equation of state parameter and limits on early dark energy contribution. The natural following step of this work consists of investigating the evolution of linear perturbations of the field and of matter fluids in this scenario.

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